A CLOSED FORMULA FOR SUBEXPONENTIAL CONSTANTS IN THE MULTILINEAR BOHNENBLUST–HILLE INEQUALITY

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ABSTRACT. For the scalar field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, the multilinear Bohnenblust–Hille inequality asserts that there exists a sequence of positive scalars $\left(C_{\mathbb{K},m}\right)_{m=1}^{\infty}$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N}\left|U(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq C_{\mathbb{K},m}\sup_{z_1,\ldots,z_m\in\mathbb{D}^N}\left|U(z_1,\ldots,z_m)\right|$$

for all m-linear form $U:\mathbb{K}^N\times\cdots\times\mathbb{K}^N\to\mathbb{K}$ and every positive integer N, where $(e_i)_{i=1}^N$ denotes the canonical basis of $\mathbb{K}^N$ and $\mathbb{D}^N$ represents the open unit polydisk in $\mathbb{K}^N$. Since its proof in 1931, the estimates for $C_{\mathbb{K},m}$ have been improved in various papers. In 2012 it was shown that there exist constants $\left(C_{\mathbb{K},m}\right)_{m=1}^{\infty}$ with subexponential growth satisfying the Bohnenblust-Hille inequality. However, these constants were obtained via a complicated recursive formula. In this paper, among other results, we obtain a closed (non-recursive) formula for these constants with subexponential growth.

1. Introduction

The complex multilinear Bohnenblust–Hille inequality asserts that for every positive integer $m\geq 1$ there exists a sequence of positive scalars $C_{\mathbb{K},m}\geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N}\left|U(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq C_{\mathbb{K},m}\sup_{z_1,\ldots,z_m\in\mathbb{D}^N}\left|U(z_1,\ldots,z_m)\right| \tag{1.1}$$

for all m-linear form $U:\mathbb{K}^N\times\cdots\times\mathbb{K}^N\to\mathbb{K}$ and every positive integer N, where $(e_i)_{i=1}^N$ is the canonical basis of $\mathbb{K}^N$ and $\mathbb{D}^N$ is the open unit polydisk in $\mathbb{K}^N$. This inequality was overlooked for some decades but it was rediscovered some years ago and, since then, several works and applications have appeared (see [3, 4, 5, 6, 7, 10, 11]). It is well-known (since the original proof of H.F. Bohnenblust and E. Hille) that the power $\frac{2m}{m+1}$ is sharp; on the other hand the optimal values of the constants $C_{\mathbb{K},m}$ are not known. In the case of real scalars the Bohnenblust–Hille inequality is also valid, but with different constants. In fact it is known that in the real case

$$C_{\mathbb{R},2}=\sqrt{2}$$

is optimal (see [7]) and, in the complex case,

$$C_{\mathbb{C},2}\leq\frac{2}{\sqrt{\pi}}.$$

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The estimates for these constants are becoming more accurate along the time. For the complex case we have:

- $C_{\mathbb{C},m} \leq m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$ (1931 - Bohnenblust and Hille [1]),
- $C_{\mathbb{C},m} \leq 2^{\frac{m-1}{2}}$ (70's - Kaijser [9] and Davie [2]),
- $C_{\mathbb{C},m} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ (1995 - Queffélec [12]).

Although the optimal constants $C_{\mathbb{K},m}$ are not known, some recent papers have investigated their asymptotical growth (see [6, 10]).Very recently, quite better estimates, with a surprising subexponential growth, were obtained in [6, 11] but the recursive way that these constants were obtained make the presentation of a closed formula a quite difficult task. One of the main goals of this paper is to present a closed formula for the constants with subexponential growth obtained in [6, 11].

2. First remarks

We begin by recalling the Khinchin inequality:

For any $p > 0$, there are constants $A_p, B_p > 0$ such that

$$A_p \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{n=1}^{\infty} a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}. \tag{2.1}$$

regardless of the $(a_n)_{n=1}^{\infty} \in l_2$. Above, r_n represents the n-th Rademacher function.

From [8] we know that the best values of A_p are

$$A_p = \begin{cases} \sqrt{2} \left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p}, \text{ if } p > p_0 \\ 2^{\frac{1}{2} - \frac{1}{p}}, \text{ if } p < p_0, \end{cases} \tag{2.2}$$

where Γ denotes the Gamma Function and $1 < p_0 < 2$ is so that

$$\Gamma\left(\frac{p_0+1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Numerical calculations estimate

$$p_0 \approx 1.847.$$

The following result appears in [10]:

Theorem 1. *For all positive integers n,*

$$C_{\mathbb{R},2} = 2^{\frac{1}{2}},$$
$$C_{\mathbb{R},3} = 2^{\frac{5}{6}}$$

and

$$C_{\mathbb{R},n} = 2^{\frac{1}{2}} \left(\frac{C_{\mathbb{R},n-2}}{A^2_{\frac{2n-4}{n-1}}} \right)^{\frac{n-2}{n}} \quad \text{for } n > 3.$$

In particular, if $2 \leq n \leq 14$

$$C_{\mathbb{R},n} = 2^{\frac{n^2+6n-8}{8n}}, \text{ if } n \text{ is even}$$
$$C_{\mathbb{R},n} = 2^{\frac{n^2+6n-7}{8n}}, \text{ if } n \text{ is odd.}$$

The above theorem allows to obtain a closed formula for the constants. It is shown in [10] that for an even positive integer $n > 14$,

$$C_{\mathbb{R},n} = 2^{\frac{n+2}{8}} r_n,$$

for a certain r_n for which numerical computations show that it tends to a number close to 1.44. The formula for r_n from [10] contains a slight imprecision which affects some decimals of the first constants. Below we show a correct formula for r_n.

Proposition 1. *If $n > 14$ is even, then*

$$C_{\mathbb{R},n} = 2^{\frac{n+2}{8}} r_n,$$

with

$$r_n = \frac{\pi^{\frac{(n+14)(n-14)}{8n}}}{2^{\frac{(n+12)(n-14)-24}{4n}} \cdot \left[\prod_{k=7}^{\frac{n-2}{2}} \left(\Gamma\left(\frac{6k+1}{4k+2}\right)\right)^{2k+1} \right]^{\frac{1}{n}}}. \tag{2.3}$$

Proof. Using the estimates from Theorem *1 we have*

$$C_{\mathbb{R},4} = 2^{\frac{1}{2}} \left(\frac{C_{\mathbb{R},2}}{A^2_{\frac{4}{3}}} \right)^{\frac{2}{4}}$$

$$C_{\mathbb{R},6} = 2^{\frac{1}{2}} \left(\frac{2^{\frac{1}{2}} \left(\frac{C_{\mathbb{R},2}}{A^2_{\frac{4}{3}}} \right)^{\frac{2}{4}}}{A^2_{\frac{8}{5}}} \right)^{\frac{4}{6}} = \frac{\left(2^{\frac{1}{2}+\frac{1}{2}\cdot\frac{4}{6}}\right) (C_{\mathbb{R},2})^{\frac{2}{4}\cdot\frac{4}{6}}}{\left(A^2_{\frac{4}{3}}\right)^{\frac{2}{4}\cdot\frac{4}{6}} \left(A^2_{\frac{8}{5}}\right)^{\frac{4}{6}}}$$

$$C_{\mathbb{R},8} = \frac{2^{\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{4}{6}\right)\frac{6}{8}} (C_{\mathbb{R},2})^{\frac{2}{4}\cdot\frac{4}{6}\cdot\frac{6}{8}}}{\left(A^2_{\frac{4}{3}}\right)^{\frac{2}{4}\cdot\frac{4}{6}\cdot\frac{6}{8}} \left(A^2_{\frac{8}{5}}\right)^{\frac{4}{6}\cdot\frac{6}{8}} \left(A^2_{\frac{12}{7}}\right)^{\frac{6}{8}}}$$

and so on. Hence

$$C_{\mathbb{R},n} = \frac{d_n}{s_n} \tag{2.4}$$

with

$$s_n = \left(A^2_{\frac{4}{3}}\right)^{\frac{2}{4}\cdot\frac{4}{6}\cdots\frac{n-2}{n}} \left(A^2_{\frac{8}{5}}\right)^{\frac{4}{6}\cdot\frac{6}{8}\cdots\frac{n-2}{n}} \left(A^2_{\frac{12}{7}}\right)^{\frac{6}{8}\cdot\frac{8}{10}\cdots\frac{n-2}{n}} \cdots \left(A^2_{\frac{2n-4}{n-1}}\right)^{\frac{n-2}{n}}$$

and

$$d_n = 2^{\frac{1}{2}+\frac{1}{2}\left(\frac{n-2}{n}\right)+\frac{1}{2}\left(\frac{n-4}{n}\right)+\frac{1}{2}\left(\frac{n-6}{n}\right)+\ldots+\frac{1}{2}\left(\frac{n-(n-4)}{n}\right)} \sqrt{2}^{\frac{2}{n}}.$$

For $p = \frac{2n-4}{n-1}$ and $2 \leq n \leq 14$, we have $p < 1.847$. So

$$A_p = 2^{\frac{1}{2}-\frac{1}{p}}$$

and, for $n > 14$, we have $p > p_0$ and

$$A_p = 2^{\frac{1}{2}} \left(\frac{\Gamma\left((p+1)/2\right)}{\sqrt{\pi}} \right)^{\frac{1}{p}}.$$

We thus have

$$
\begin{aligned}
s_n &= \left(A^2_{\frac{4}{3}}\right)^{\frac{2}{4}\cdot\frac{4}{6}\cdots\frac{n-2}{n}} \left(A^2_{\frac{8}{5}}\right)^{\frac{4}{6}\cdot\frac{6}{8}\cdots\frac{n-2}{n}} \dots \left(A^2_{\frac{24}{13}}\right)^{\frac{12}{14}\cdot\frac{14}{16}\cdots\frac{n-2}{n}} \left(A^2_{\frac{28}{15}}\right)^{\frac{14}{16}\cdot\frac{16}{18}\cdots\frac{n-2}{n}} \dots \left(A^2_{\frac{2n-4}{n-1}}\right)^{\frac{n-2}{n}} \\
&= \left(2^{-\frac{1}{4}}\right)^{\frac{2}{\left(\frac{n}{2}\right)}} \left(2^{-\frac{1}{8}}\right)^{\frac{4}{\left(\frac{n}{2}\right)}} \dots \left(2^{-\frac{1}{24}}\right)^{\frac{12}{\left(\frac{n}{2}\right)}} \times \\
&\times \left(\sqrt{2}\left(\frac{\Gamma\left(\frac{43}{30}\right)}{\sqrt{\pi}}\right)^{\frac{15}{28}}\right)^{\frac{14}{\frac{n}{2}}} \dots \left(\sqrt{2}\left(\frac{\Gamma\left(\left(\frac{2n-4}{n-1}+1\right)/2\right)}{\sqrt{\pi}}\right)^{\frac{1}{p}}\right)^{\frac{n-2}{n}} \\
&= \left(2^{-\frac{6}{n}}\right) \left(\sqrt{2}\left(\frac{\Gamma\left(\frac{43}{30}\right)}{\sqrt{\pi}}\right)^{\frac{15}{28}}\right)^{\frac{14}{\frac{n}{2}}} \dots \left(\sqrt{2}\left(\frac{\Gamma\left(\left(\frac{2n-4}{n-1}+1\right)/2\right)}{\sqrt{\pi}}\right)^{\frac{1}{p}}\right)^{\frac{n-2}{n}} \\
&= 2^{\frac{(n+12)(n-14)-24}{4n}} \left(\prod_{k=7}^{\frac{n-2}{2}} \left(\frac{\Gamma\left(\frac{6k+1}{4k+2}\right)}{\sqrt{\pi}}\right)^{2k+1}\right)^{\frac{1}{n}} \\
&= 2^{\frac{(n+12)(n-14)-24}{4n}} \left(\prod_{k=7}^{\frac{n-2}{2}} \left(\Gamma\left(\frac{6k+1}{4k+2}\right)\right)^{2k+1}\right)^{\frac{1}{n}} \left(\pi^{\frac{(n+14)(n-14)}{8n}}\right)^{-1} \\
&= \frac{1}{r_n}.
\end{aligned}
$$

On the other hand a simple calculation shows that

$$d_n = 2^{\frac{n+2}{8}}$$

and from (2.4) we obtain

$$C_{\mathbb{R},n} = 2^{\frac{n+2}{8}} r_n.$$

□

Below we compare the values of the r_n from (2.3) and the r_n from [10]:

n	r_n (2.3)	r_n ([10])
30	1.387	1.375
50	1. 404	1.397
100	1. 420	1.416
250	1. 431	1.429
500	1.435	1.434
1,000	1. 4374	1.4371
10,000	1.43989	1.43986
100,000	1.44021	1.44021

Hence, although the formulas for r_n are different its values are very close and, as in [10], numerical estimates indicate that

$$\lim_{n\to\infty} r_n \approx 1.44025.$$

We conjecture that

$$\lim_{n\to\infty} r_n = \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}},$$

where γ denotes the Euler constant.

3. Main results

In [6] it was shown that there is a constant D (probably very close to 1.44) so that the sequence $(C_n)_{n=1}^{\infty}$ given by

$$\begin{aligned} C_{2n} &= DC_n \\ C_{2n+1} &= D\left(C_n\right)^{\frac{2n}{4n+2}}\left(C_{n+1}\right)^{\frac{2n+2}{4n+2}}, \end{aligned}$$

with

$$C_1 = 1 \text{ and } C_2 = \sqrt{2}$$

in the real case and

$$C_1 = 1 \text{ and } C_2 = \frac{2}{\sqrt{\pi}}$$

in the complex case, satisfies the Bohnenblust–Hille inequality and, moreover, this sequence is subexponential. From now on C_n will denote the numbers given by the above formulas.

In this section we present a closed formula for these constants. Given a positive integer n, it is plain that it can be written (in an unique way) as

$$n = 2^k - l, \tag{3.1}$$

where k is the smaller positive integer such that $2^k \geq n$ and $0 \leq l < 2^{k-1}$.

Theorem 2. *If $n \geq 3$ is written as (3.1), then*

$$C_n = D^{k-1} C_2^{\frac{n-l}{n}}, \text{ if } l \leq 2^{k-2} \tag{3.2}$$

and

$$C_n = D^{\frac{n(k-1)+2^{k-1}-2l}{n}} C_2^{\frac{2^{k-1}}{n}}, \text{ if } 2^{k-2} < l < 2^{k-1} \tag{3.3}$$

where

$$\begin{aligned} C_2 &= \sqrt{2}, \text{ for real scalars} \\ C_2 &= \frac{2}{\sqrt{\pi}} \text{ for complex scalars} \end{aligned}$$

Proof. Since $n \geq 3$, note that $k \geq 2$.

We proceed by induction. Suppose the result valid for all $m \leq n$.

Let

$$n + 1 = 2^k - l$$

with l and k so that k is the smaller positive integer such that $2^k \geq n+1$ and $0 \leq l < 2^{k-1}$.

- **First Case:** l is even.

In this case $n+1$ is even and

$$C_{n+1} = D\left(C_{\frac{n+1}{2}}\right) \tag{3.4}$$

with

$$\frac{n+1}{2} = 2^{k-1} - \frac{l}{2}.$$

By induction hypothesis, the result is valid for $C_{\frac{n+1}{2}}$. We have two possible subcases for $\frac{l}{2}$:

Subcase 1a -

$$\frac{l}{2} \leq 2^{(k-1)-2} = 2^{k-3}. \tag{3.5}$$

Subcase 1b -

$$2^{(k-1)-2} < \frac{l}{2} < 2^{(k-1)-1},$$

i.e.,

$$2^{k-3} < \frac{l}{2} < 2^{k-2}. \tag{3.6}$$

If (3.5) occurs, note that $l \leq 2^{k-2}$, from (3.4) we have

$$\begin{aligned} C_{n+1} &= D\left(D^{k-2} C_2^{\frac{n+1-l}{n+1}}\right) \\ &= D^{k-1} C_2^{\frac{(n+1)-l}{n+1}}, \end{aligned}$$

and this is what we need.

If (3.6) occurs, note that $2^{k-2} < l < 2^{k-1}$. From (3.4) we have

$$\begin{aligned} C_{n+1} &= D\left(C_{\frac{n+1}{2}}\right) \\ &= D\left(D^{\frac{(n+1)(k-2)+2^{k-1}-2l}{n+1}} C_2^{\frac{2^{k-1}}{n+1}}\right) \\ &= D^{\frac{(n+1)(k-1)+2^{k-1}-2l}{n+1}} C_2^{\frac{2^{k-1}}{n+1}} \end{aligned}$$

and again we get the desired result.

- **Second Case:** l is odd.

  In this case $n+1$ is odd and

$$\begin{aligned} C_{n+1} &= D\left(\left(C_{\frac{(n+1)-1}{2}}\right)^{\frac{\frac{(n+1)-1}{2}}{n+1}} \left(C_{\frac{(n+1)+1}{2}}\right)^{\frac{\frac{(n+1)+1}{2}}{n+1}}\right) \\ &= D\left(\left(C_{\frac{n}{2}}\right)^{\frac{\frac{n}{2}}{(n+1)}} \left(C_{\frac{n+2}{2}}\right)^{\frac{\frac{n+2}{2}}{(n+1)}}\right) \end{aligned} \tag{3.7}$$

  Since $n+1 = 2^k - l$, we have

$$n = 2^k - (l+1)\,,$$

  and

$$n+2 = 2^k - (l-1)\,.$$

  Since $0 \leq l < 2^{k-1}$, and l is odd, then

$$0 \leq l+1 < 2^{k-1}, \text{ or } l+1 = 2^{k-1}$$

and we have two subcases:

Subcase 2a -

$$n = 2^{k-1} - 0, \text{ and } n+2 = 2^k - (l-1), \tag{3.8}$$

with

$$l = 2^{k-1} - 1$$

Subcase 2b -

$$n = 2^k - (l+1), \text{ and } n+2 = 2^k - (l-1), \tag{3.9}$$

with

$$l < 2^{k-1} - 1.$$

If (3.8) holds, then $l = 2^{k-1} - 1$. Since

$$\frac{n}{2} = 2^{k-2},$$

then $C_{\frac{n}{2}}$ is of the form (3.2) and, since

$$\frac{n+2}{2} = 2^{k-1} - \frac{(l-1)}{2} \text{ and } 2^{k-3} < \frac{l-1}{2} < 2^{k-2},$$

then $C_{\frac{n+2}{2}}$ is of the form (3.3). We this have

$$C_{\frac{n}{2}} = D^{(k-2)-1} C_2$$

and

$$C_{\frac{n+2}{2}} = D^{\frac{\frac{n+2}{2}(k-2)+2^{(k-1)-1}-2\left(\frac{l-1}{2}\right)}{\frac{n+2}{2}}} C_2^{\frac{2^{(k-1)-1}}{\frac{n+2}{2}}}.$$

From (3.7), we get

$$\begin{aligned} C_{n+1} &= D\left(C_{\frac{n}{2}}\right)^{\frac{\frac{n}{2}}{(n+1)}} \left(C_{\frac{n+2}{2}}\right)^{\frac{\frac{n+2}{2}}{(n+1)}} \\ &= D\left(D^{k-3} C_2\right)^{\frac{\frac{n}{2}}{(n+1)}} \left(D^{\frac{\frac{n+2}{2}(k-2) + \frac{2^{k-1}}{2} - 2\left(\frac{l-1}{2}\right)}{\frac{n+2}{2}}} C_2^{\frac{2^{k-2}}{\frac{n+2}{2}}} \right)^{\frac{\frac{n+2}{2}}{(n+1)}} \\ &= D^{\frac{(n+1)(k-1)+2^{k-1}-2l}{n+1}} C_2^{\frac{2^{k-1}}{n+1}} \end{aligned}$$

and we have the desired result.

In the case that (3.9) holds, we have

$$l+1 < 2^{k-1},$$

$$\frac{n}{2} = 2^{k-1} - \frac{(l+1)}{2}$$

and

$$\frac{n+2}{2} = 2^{k-1} - \frac{(l-1)}{2}.$$

We have three sub-subcases:

Sub-subcase 2ba -

$$2^{k-2} < l+1 < 2^{k-1}, \text{ and } 2^{k-2} < l-1 < 2^{k-1} \tag{3.10}$$

Sub-subcase 2bb -

$$2^{k-2} < l+1 < 2^{k-1}, \text{ and } l-1 = 2^{k-2} \tag{3.11}$$

Sub-subcase 2bc -

$$l-1 < l+1 \leq 2^{k-2}. \tag{3.12}$$

If (3.10) holds, note that

$$2^{k-2} < l-1 < l < l+1 < 2^{k-1},$$

and this C_{n+1} is of the form (3.3). Therefore

$$2^{k-3} < \frac{l+1}{2} < 2^{k-2}, \text{ and } 2^{k-3} < \frac{l-1}{2} < 2^{k-2}$$

and this $C_{\frac{n}{2}}$ and $C_{\frac{n+2}{2}}$ are written in the form (3.3); now, from (3.7) we have

$$\begin{aligned}
C_{n+1} &= D\left(C_{\frac{n}{2}}\right)^{\frac{\frac{n}{2}}{(n+1)}}\left(C_{\frac{n+2}{2}}\right)^{\frac{\frac{n+2}{2}}{(n+1)}} \\
&= D\left(D^{\frac{\frac{n}{2}(k-2)+2^{k-2}-2\left(\frac{l+1}{2}\right)}{\frac{n}{2}}} C_2^{\frac{2^{k-2}}{\frac{n}{2}}}\right)^{\frac{\frac{n}{2}}{n+1}} \left(D^{\frac{\frac{n+2}{2}(k-2)+2^{k-2}-2\left(\frac{l-1}{2}\right)}{\frac{n+2}{2}}} C_2^{\frac{2^{k-2}}{\frac{n+2}{2}}}\right)^{\frac{\frac{n+2}{2}}{n+1}} \\
&= D^{\frac{(n+1)(k+1)+2^{k-1}-2l}{n+1}} C_2^{\frac{2^{k-1}}{n+1}}.
\end{aligned}$$

If (3.11) holds, note that

$$2^{k-2} = l-1 < l < l+1 < 2^{k-1},$$

and we need to obtain a formula like (3.3). Since

$$2^{k-3} < \frac{l+1}{2} < 2^{k-2}, \text{ and } \frac{l-1}{2} = 2^{k-3}$$

then $C_{\frac{n}{2}}$ is represented by (3.3) and $C_{\frac{n+2}{2}}$ *is* of the form (3.2). So, from (3.7), we have

$$\begin{aligned}
C_{n+1} &= D\left(D^{\frac{\frac{n}{2}(k-2)+2^{k-2}-2\left(\frac{l+1}{2}\right)}{\frac{n}{2}}} C_2^{\frac{2^{k-2}}{\frac{n}{2}}}\right)^{\frac{\frac{n}{2}}{n+1}} \left(D^{k-2} C_2^{\frac{\frac{n+2}{2}-2^{k-3}}{\frac{n+2}{2}}}\right)^{\frac{\frac{n+2}{2}}{n+1}} \\
&= D.D^{\frac{\frac{n}{2}(k-2)+2^{k-2}-(l+1)}{n+1}}.D^{\frac{(k-2)\left(\frac{n+2}{2}\right)}{n+1}}.C_2^{\frac{2^{k-2}}{n+1}}.C_2^{\frac{\frac{n+2}{2}-2^{k-3}}{n+1}} \\
&= D^{\frac{(n+1)+(k-2)(n+1)+2^{k-2}-2^{k-2}-2}{n+1}}.C_2^{\frac{2^{k-3}+\frac{n+2}{2}}{n+1}}
\end{aligned}$$

Since

$$2^{k-1} - 2l = 2^{k-1} - 2\left(2^{k-2}+1\right) = -2,$$

and

$$\frac{n+2}{2} = 2^{k-1} - \left(\frac{l-1}{2}\right) = 2^{k-1} - 2^{k-3},$$

then

$$C_{n+1} = D^{\frac{(n+1)(k-1)+2^{k-1}-2l}{n+1}} C_2^{\frac{2^{k-1}}{n+1}}.$$

Finally, if we have (3.12), note that

$$l-1 < l < l+1 \leq 2^{k-2},$$

and then C_{n+1} must be of the form (3.2). Hence

$$\frac{l-1}{2} < \frac{l+1}{2} \leq 2^{k-3},$$

and $C_{\frac{n}{2}}$ and $C_{\frac{n+2}{2}}$ are written in the form of (3.2). Thus, again using (3.7) we have

$$\begin{aligned}
C_{n+1} &= D\left(D^{k-2}C_2^{\frac{\frac{n}{2}-\frac{l+1}{2}}{\frac{n}{2}}}\right)^{\frac{\frac{n}{2}}{n+1}}\left(D^{k-2}C_2^{\frac{\frac{n+2}{2}-\frac{l-1}{2}}{\frac{n+2}{2}}}\right)^{\frac{\frac{n+2}{2}}{n+1}} \\
&= D.D^{\frac{(k-2)\frac{n}{2}}{n+1}}.D^{\frac{(k-2)\frac{n+2}{2}}{n+1}}.C_2^{\frac{\frac{n}{2}-\frac{l+1}{2}}{n+1}}.C_2^{\frac{\frac{n+2}{2}-\frac{l-1}{2}}{n+1}} \\
&= D^{\frac{(n+1)+(k-2)\left(\frac{n}{2}+\frac{n+2}{2}\right)}{n+1}}C_2^{\frac{2^{k-1}-\left(\frac{l+1}{2}\right)-\left(\frac{l+1}{2}\right)}{n+1}}C_2^{\frac{2^{k-1}-\left(\frac{l-1}{2}\right)-\left(\frac{l-1}{2}\right)}{n+1}} \\
&= D^{k-1}C_2^{\frac{(n+1)-l}{n+1}},
\end{aligned}$$

and the proof is done. □

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